

Reflection symmetry in mean-field replica-symmetric spin glasses

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Abstract. The role of reflection symmetry breaking for the character of the appearance of replica symmetric spin glass state is investigated. We establish the following symmetry rule for classical systems with one order parameter in the replica symmetric mean field approximation. If in the pure system the transition to the ordered phase is of the second order, then in the corresponding random system the glass regime appears as a result of a phase transition; if the transition in the pure system is of the first order, then glass and ordered regimes grow continuously in the random system on cooling.

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The crucial role of the reflection symmetry for the character of phase transition in nonrandom mean-field (MF) models is well known (see, e.g., the textbook [1]). Generally speaking the presence of the terms without reflection symmetry causes the first order phase transition, while in the absence of such terms the transition is of the second order. Usually this result is obtained in the frame of the phenomenological approach based on the Ginsburg-Landau (GL) effective Hamiltonian that can be easily obtained for any Hamiltonian through the Hubbard-Stratanovich identity for the partition function. On a slightly "more microscopic level" the statement about the order of phase transition can be demonstrated considering the MF one-site equations as is done below (for continuous case see the Eqs.(11-13) of [2]).

Let us consider a system of particles on lattice sites i, j with Hamiltonian

$$H = -\frac{1}{2} \sum_{i \neq j} J_{ij} \hat{U}_i \hat{U}_j, \quad (1)$$

where \hat{U} is a diagonal operator with $\text{Tr} \hat{U} = 0$, the interactions J_{ij} being such that the MF approximation gives exact solution. For example, J_{ij} may be random exchange interactions with Gaussian probability distribution

$$P(J_{ij}) = \frac{1}{\sqrt{2\pi J}} \exp \left[-\frac{(J_{ij} - J_0)^2}{2J^2} \right] \quad (2)$$

with $J = \tilde{J}/\sqrt{N}$, $J_0 = \tilde{J}_0/N$. In nonrandom pure case $J = 0$. Let us construct GL Hamiltonian as an integral on

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fluctuations φ . Using the Hubbard-Stratanovich identity the partition sum can be written in the following form:

$$\begin{aligned} Z = \text{Tr} \exp \left(\frac{1}{2} \beta \sum_{ij} \hat{U}_i J_{ij} \hat{U}_j \right) = & \frac{[\det(\beta J)^{-1}]^{1/2}}{[2\pi]^{N/2}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{i=1}^N d\varphi_i \times \\ & \exp \left\{ -\frac{1}{2} \sum_{ij} \varphi_i (\beta J)_{ij}^{-1} \varphi_j + \sum_i \ln \text{Tr} \exp [\varphi_i \hat{U}_i] \right\} = \\ & C \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{i=1}^N d\varphi_i \exp \{ -\beta H_{eff} \} \quad (3) \end{aligned}$$

The effective Hamiltonian contains the powers n of the fluctuations φ for which $\text{Tr}(\hat{U}^n)$ is nonzero.

When dealing with random MF systems the role of cubic term is discussed usually in connection with the replica symmetry breaking (RSB) solution of the equations for glass order parameters. The absence of reflection symmetry causing cubic terms in GL Hamiltonian for regular nonrandom system results in a special form of RSB free energy functional for random case, too. This form gives the discontinuity for the order parameter, the stability of the first stage RSB solution and other specific features (see, for example, Refs. [3,4,5]). However, in this paper we accent our attention on the replica symmetric (RS) approach. We investigate the role of the reflection symmetry for the behavior of RS solution for spin-glass-like MF systems with one regular order parameter in the Hamiltonian and formulate a kind of symmetry rule for the type

of the growing of glass regime: if in the nonrandom (pure) system the transition to the ordered phase is of the second order, then in the corresponding random system the glass regime appears as a result of a phase transition; if the transition in the pure system is of the first order, then in the random system the glass regime grows continuously on cooling. In fact, one can imagine that both the first order phase transition in nonrandom systems as well as the continuous growing of the glass regime in random systems are caused by some kind of internal fields appearing due to the algebra of operators \hat{U} . Some previous results can be found in [6]. It is well known that a number of magnetic compounds with phase transition to spin-glass state behaves qualitatively as the Sherrington-Kirkpatrick (SK) model [7] (without φ^3 term). We would like to bring the readers attention to the fact that the case with the continuous growing of the glass regime (with φ^3 term) also is a reality and exists in some mixed crystals [8,9].

Now let us consider the system (1)-(2). Using replica approach (see, e.g. [7]) we can write the free energy in the form

$$\begin{aligned} \langle F \rangle_J / NkT = & - \lim_{n \rightarrow 0} \frac{1}{n} \max \left\{ - \sum_{\alpha} \frac{(x^{\alpha})^2}{2} - \right. \\ & \sum_{\alpha} \frac{(w^{\alpha})^2}{2} - \sum_{\alpha > \beta} \frac{(y^{\alpha\beta})^2}{2} + \\ & \ln \text{Tr}_{\{U^{\alpha}\}} \exp \left[\sum_{\alpha} x^{\alpha} \sqrt{\frac{\tilde{J}_0}{kT}} U^{\alpha} + \sum_{\alpha} w^{\alpha} \frac{1}{\sqrt{2}} t (U^{\alpha})^2 + \right. \\ & \left. \left. \sum_{\alpha > \beta} y^{\alpha\beta} t U^{\alpha} U^{\beta} \right] \right\} \quad (4) \end{aligned}$$

where

$$\begin{aligned} m^{\alpha} &= (x^{\alpha})^{\text{extr}} / \sqrt{\frac{\tilde{J}_0}{kT}} = \langle U^{\alpha} \rangle_{\text{eff}}; \\ q^{\alpha\beta} &= (y^{\alpha\beta})^{\text{extr}} / t = \langle U^{\alpha} U^{\beta} \rangle_{\text{eff}}; \\ p^{\alpha} &= (w^{\alpha})^{\text{extr}} \sqrt{2} / t = \langle (U^{\alpha})^2 \rangle_{\text{eff}}, \end{aligned} \quad (5)$$

as follows from the saddle point equations. Here $t = \tilde{J}/kT$ and averaging is performed with the effective Hamiltonian \mathcal{H}_{eff} :

$$-\mathcal{H}_{\text{eff}} = \sum_{\alpha} \frac{\tilde{J}_0}{kT} m^{\alpha} U^{\alpha} + \sum_{\alpha} \frac{t^2}{2} p^{\alpha} (U^{\alpha})^2 + \sum_{\alpha > \beta} t^2 q^{\alpha\beta} U^{\alpha} U^{\beta}$$

In the RS approximation the free energy has the form:

$$\begin{aligned} F = & -NkT \left\{ - \left(\frac{\tilde{J}_0}{kT} \right) \frac{m^2}{2} + t^2 \frac{q^2}{4} - t^2 \frac{p^2}{4} + \right. \\ & \left. \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp \left(-\frac{z^2}{2} \right) \ln \text{Tr} \left[\exp(\hat{\theta}) \right] \right\} \quad (6) \end{aligned}$$

Here

$$\hat{\theta} = \left[zt\sqrt{q} + m \left(\frac{\tilde{J}_0}{kT} \right) \right] \hat{U} + t^2 \frac{p-q}{2} \hat{U}^2.$$

The order parameters are: m is the regular order parameter (an analog of magnetic moment in spin glasses), q is the glass order parameter and p is an auxiliary order parameter. The extremum conditions for the free energy (6) give the following equations for these order parameters:

$$m = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp \left(-\frac{z^2}{2} \right) \frac{\text{Tr} [\hat{U} \exp(\hat{\theta})]}{\text{Tr} [\exp(\hat{\theta})]} \quad (7)$$

$$q = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp \left(-\frac{z^2}{2} \right) \left\{ \frac{\text{Tr} [\hat{U} \exp(\hat{\theta})]}{\text{Tr} [\exp(\hat{\theta})]} \right\}^2 \quad (8)$$

$$p = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp \left(-\frac{z^2}{2} \right) \frac{\text{Tr} [\hat{U}^2 \exp(\hat{\theta})]}{\text{Tr} [\exp(\hat{\theta})]} \quad (9)$$

In general case the order parameter p is not independent from the others. For Ising spin glass [7] $\hat{U} = \hat{S}$, $(\hat{U})^2 = (\hat{S})^2 = 1 = p$ and Eq. (9) reduces to an identity. For the quadrupolar glass with $J = 1$ [9] we have $\hat{U} = \hat{Q} = 3\hat{J}_z^2 - 2$, so that $p = 2 - m$. However, for the quadrupolar glass with $J = 2$ [10] p is independent order parameter: the subalgebra contains the operators \hat{Q} , $(\hat{Q})^2$ and the unit matrix \hat{E} , so that only $(\hat{Q})^3$ is not independent and can be represented as a linear combination of \hat{Q} , $(\hat{Q})^2$ and \hat{E} . In the case of $S = 1$ spin glass [11] we have $S_z = 0, \pm 1$, p is independent parameter and has the physical meaning defined by the equality $\hat{S}^2 = 1/3(2 + \hat{Q})$. Now $p = (1/3)(2 + m_2)$, where m_2 is the quadrupolar regular order parameter.

In the random case $J_0 = 0$ the high temperature expansion of the equations (7) - (9) has the form:

$$\begin{aligned} m = & \frac{t^2}{2} A_3 p + \frac{t^4}{8} p^2 (A_5 - 2A_2 A_3) - \frac{t^4}{2} A_2 A_3 q^2 - \\ & - \frac{t^4}{2} A_2 A_3 p q + \dots \end{aligned} \quad (10)$$

$$q = t^2 A_2^2 q + \frac{t^4}{2} A_3^2 p^2 + \frac{t^4}{2} q^2 (A_3^2 - 4A_2^3) + t^4 A_2 A_4 q p + \dots \quad (11)$$

$$\begin{aligned} p = & A_2 + \frac{t^2}{2} p (A_4 - A_2^2) + \frac{t^4}{8} p^2 (A_6 - 3A_2 A_4 + 2A_2^3) - \\ & - \frac{t^4}{2} q^2 A_2 (A - 4 - A_2^2) - \frac{t^4}{2} p q A_3^2 + \dots \end{aligned} \quad (12)$$

If $m = 0, q = 0$ and $p \neq 0$ then

$$\text{Tr}[\hat{U} \exp(\hat{\theta})] = \text{Tr}[\hat{U} \exp(t^2 p \hat{U}^2 / 2)] \quad (13)$$

is a sum of $A_n = \text{Tr}(\hat{U}^n)$ with odd n only. So if $A_{(2k+1)} = 0$ then Eqs.(7) - (9) have the solution $m = 0, q = 0$. If $A_{(2k+1)} \neq 0$ Eqs.(7) - (9) have no trivial solution $m = 0, q = 0$ at any temperature. The high temperature expansion gives explicitly $m \sim t^2 A_3, q \sim t^4 A_3^2$. The glass order parameter grows continuously on cooling.

In the case $A_{(2k+1)} = 0$ one can obtain from (8) the bifurcation point T_c^b where the nontrivial solution for the glass order parameter q appears under the condition $m = 0$. The equations for T_c^b have the form:

$$p(t_c) = 1/t_c; \quad t_c \equiv \tilde{J}/kT_c^b$$

$$p(t_c) = \text{Tr}[\hat{U}^2 \exp(t_c^2 p(t_c) \hat{U}^2/2)].$$

The replica symmetric solution is stable unless the replicon mode energy λ is nonzero. For our model we have:

$$\lambda_{repl} = 2 - 2t^2 \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \left\{ \frac{\text{Tr}[\hat{U}^2 \exp(\hat{\theta})]}{\text{Tr}[\exp(\hat{\theta})]} - \left[\frac{\text{Tr}[\hat{U} \exp(\hat{\theta})]}{\text{Tr}[\exp(\hat{\theta})]} \right]^2 \right\}. \quad (14)$$

The character of the replica symmetry breaking is defined by the term $\alpha_3(\delta q_{\alpha\beta})^3$ in the RSB free energy (4) expansion near RS free energy (6). We have

$$\alpha_3 = -t^6 \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \left[\frac{\text{Tr}[\hat{U}^3 \exp(\hat{\theta})]}{\text{Tr}[\exp(\hat{\theta})]} \right]^2 - 3t^6 q \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \left[\frac{\text{Tr}[\hat{U}^2 \exp(\hat{\theta})]}{\text{Tr}[\exp(\hat{\theta})]} \right]^2 + 2t^6 q^3.$$

If all $A_{2n+1} = 0$ then $\text{Tr}[\hat{U}^3 \exp(\hat{\theta})] = 0$ and $q = 0$ at $T = T_c$ so that $\alpha_3 = 0$ and at T_c the full replica symmetry breaking according to the Parisi scheme takes place. If $A_{2n+1} \neq 0$ then $\alpha_3 \neq 0$ and the first stage RSB solution is stable.

In the regular case ($\tilde{J} = 0, \tilde{J}_0 \neq 0$) we have from Eqs.(7) – (9) $q \equiv m^2$ and

$$m = \frac{\text{Tr}[\hat{U} \exp(m(\frac{\tilde{J}_0}{kT}) \hat{U})]}{\text{Tr}[\exp(m(\frac{\tilde{J}_0}{kT}) \hat{U})]} \quad (15)$$

$$p = \frac{\text{Tr}[\hat{U}^2 \exp(m(\frac{\tilde{J}_0}{kT}) \hat{U})]}{\text{Tr}[\exp(m(\frac{\tilde{J}_0}{kT}) \hat{U})]} \quad (16)$$

These mean field equations have the trivial solution $m_0 = 0$ and $p_0 = \text{Tr}(\hat{U}^2)/\text{Tr}(\hat{E})$ for all temperatures. A non-trivial solution for m appears at the bifurcation point $kT_c^b/\tilde{J}_0 = p_0$. Near T_c^b the expansion of (15) – (16) gives the bifurcation equation

$$\frac{m^2}{2} A_2 \lambda_0^2 - \tau A_2 - \frac{m}{2} A_3 \lambda_0^2 - \frac{m^2}{6} A_4 \lambda_0^3 - m \tau \lambda_0 A_3 + \dots = 0 \quad (17)$$

where $\frac{\tilde{J}_0}{kT} = \lambda, \frac{\tilde{J}_0}{kT_c^b} = \lambda_0, \lambda = \lambda_0 + \tau$.

If $A_3 = 0$, the solution of (17) has the form

$$m = \sqrt{\tau \frac{A_2}{\frac{1}{2} A_2 \lambda_0^2 - \frac{1}{6} A_4 \lambda_0^3}} = \sqrt{\frac{6\tau A_2^4}{3A_2^2 - A_4}} \quad (18)$$

and we have the second order phase transition. If $A_3 \neq 0$ the solution of Eq.(15) in the vicinity of T_c^b is

$$m = -\frac{2\tau A_2^3}{A_3} \quad (19)$$

In this case the phase transition is of the first order. The transition temperature can be obtained from the comparison of free energies of ordered and disordered phases. (see, e.g. Ref.[2,12]).

So one can establish the following symmetry rule for classical systems with one (pure) order parameter in the replica-symmetric MFA. If in the pure system the transition to the ordered phase is of the second order, then in the corresponding random system the glass regime appears as a result of a phase transition; if the transition in the pure system is of the first order, then in the random system glass and ordered regimes grow smoothly on cooling.

As an example let us now consider the Hamiltonian (1) with $\hat{U} = \hat{Q} + \eta \hat{V}$, $\hat{Q} = 3\hat{J}_z^2 - 2$, $\hat{V} = \sqrt{3}(\hat{J}_x^2 - \hat{J}_y^2)$, $J = 1$, $J_z = 1, 0, -1$. This model describes random quadrupole interaction. A molecular quadrupolar moment is the second-rank tensorial operator with five independent components. In the principal axes frame only two of them remain: \hat{Q} and \hat{V} . It is easy to show that $\hat{Q}^2 = 2 - \hat{Q}$, $\hat{V}^2 = 2 + \hat{Q}$, $\hat{Q}\hat{V} = \hat{V}\hat{Q} = \hat{V}$. Here $\eta > 0$ is a tuning parameter.

$$(\hat{Q} + \eta \hat{V})^2 = 2(1 + \eta^2) + 2\eta \hat{V} + (1 - \eta^2) \hat{Q}. \quad (20)$$

The corresponding Ginzburg-Landay Hamiltonian written in terms of fluctuation fields φ_i has the form:

$$\beta H_{eff} = \frac{1}{2} \sum_{ij} \varphi_i (\beta J)_{ij}^{-1} \varphi_j - N \ln 3 - N \sum_i \left[\varphi_i^2 (1 + \eta^2) + \varphi_i^3 (\eta^2 - 1/3) + \varphi_i^4 (1 + \eta^2)^2 / 4 + \dots \right] \quad (21)$$

The RS free energy has the form

$$F = -NkT \left\{ -\left(\frac{\tilde{J}_0}{kT}\right) \frac{m_1^2}{2} + t^2 \frac{q^2}{4} - t^2 \frac{m_2^2}{4} + t^2 (1 + \eta^2)^2 - t^2 (1 + \eta^2) q + + \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \ln \psi \right\} \quad (22)$$

Here

$$\psi = \exp(-2\vartheta_1) + \exp(\vartheta_1) [\exp(\vartheta_2) + \exp(-\vartheta_2)]$$

$$\vartheta_1 = zt\sqrt{q} + m_1 \left(\frac{\tilde{J}_0}{kT} \right) + t^2(\eta^2 - 1) \left[(\eta^2 + 1) - \frac{m_2}{2} - \frac{q}{2} \right]$$

$$\vartheta_2 = \eta\sqrt{3}t^2[2(1+\eta^2)-(m_2+q)] + \eta\sqrt{3} \left[m_1 \left(\frac{\tilde{J}_0}{kT} \right) + zt\sqrt{q} \right].$$

The order parameters are $m_1 \ll \hat{Q} + \eta\hat{V} \gg$, q is the corresponding glass order parameter, the order parameter $m_2 \ll (1 - \eta^2)\hat{Q} - 2\eta\hat{V} \gg$ appears due to the operators algebra (see Eq. (20)) in close analogy with the appearing of quadrupole order parameter in the case of spin glass with $S = 1$. We use m_2 instead of p :

$$p = 2(\eta^2 + 1) - m_2.$$

The equations for the order parameters have the form:

$$m_1 = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \left[\frac{\frac{\partial\psi}{\partial\vartheta_1} + \eta\sqrt{3}\frac{\partial\psi}{\partial\vartheta_2}}{\psi} \right], \quad (23)$$

$$m_2 = - \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \left[\frac{(\eta^2 - 1)\frac{\partial\psi}{\partial\vartheta_1} + 2\eta\sqrt{3}\frac{\partial\psi}{\partial\vartheta_2}}{\psi} \right], \quad (24)$$

$$q = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \left[\frac{\frac{\partial\psi}{\partial\vartheta_1} + \eta\sqrt{3}\frac{\partial\psi}{\partial\vartheta_2}}{\psi} \right]^2, \quad (25)$$

It is easy to see that

$$(\hat{Q} + \eta\hat{V})^3 = 2(3\eta^2 - 1) + 3(1 + \eta^2)(\hat{Q} + \eta\hat{V});$$

$$\text{Tr}(\hat{Q} + \eta\hat{V})^3 = 6(3\eta^2 - 1)$$

Using these equalities and Eqs. (21) one can show that the behavior of the system under consideration is quite different in the cases $\eta = 1/\sqrt{3}$ and $\eta \neq 1/\sqrt{3}$.

Let us consider first the regular case ($\tilde{J} = 0$, $\tilde{J}_0 \neq 0$). Now we have from Eqs. (15)–(19):

$$m_1 = \frac{16}{3}\sqrt{\tau}, 3\eta^2 = 1;$$

$$m_1 = -\frac{8\tau(1 + \eta^2)^3}{3\eta^2 - 1}, 3\eta^2 - 1 \neq 0.$$

The case $\eta = 1/\sqrt{3}$ has transparent physical meaning for $\tilde{J}_0 > 0$, second order transition for spin one ferromagnet being an example. In the case $3\eta^2 - 1 \neq 0$ we obtain the first order phase transition for $\eta > 1/\sqrt{3}$ and $\tilde{J}_0 > 0$ or for $\eta < 1/\sqrt{3}$ and $\tilde{J}_0 < 0$. It should be noted that an example of such a behavior is presented by the case $\eta = 0$ describing the orientational quadrupolar ordering in $o-H_2$ or $p-D_2$ (see [2,13]). We can re write equation (15) as

$$\Phi(y, \lambda) = (y - 4\lambda) \exp(3y/2) + 2y + 4\lambda = 0 \quad (26)$$

where $y = -2m \left(\frac{\tilde{J}_0}{kT} \right)$, $\lambda = \left(\frac{\tilde{J}_0}{kT} \right)$.

The bifurcation points λ_i for $\Phi(y, \lambda)$ [13] are obtained from the condition $\left(\frac{\partial\Phi}{\partial y} \right) = 0$. We have

$$\lambda_1 = 1/2; y_1 = 0; m_1(1) = 0;$$

$$\lambda_2 = 1/2.18; y_2 = 0.7; m_1(2) = -0.8.$$

The form of the solutions can be obtained from the expansion of (26) in powers of $\tau = \lambda - \lambda_\alpha$ and $\xi = y - y_\alpha$. Because $\text{Tr}\hat{Q}^3 \neq 0$ in the vicinity of $T_1^b = \tilde{J}_0/k\lambda_1$ we have $\xi \sim \tau$. However in the vicinity of $T_2^b = 1.09T_1^b$ we have $\xi \sim \sqrt{\tau}$. The physical solution near T_2^b takes place only for $\mu > 0$, $T < T_2^b$. It is a turning point. The sharp phase transition with the jump of m is between T_1^b and T_2^b . The transition point is calculated from the comparison of energies. Other cases with $\eta \neq 1/\sqrt{3}$ can be described in a similar way.

Let us consider now the random case ($\tilde{J} \neq 0, \tilde{J}_0 = 0$). The first terms of the high temperature expansion of Eqs. (7) – (9) in the case $\hat{U} = \hat{Q} + \eta\hat{V}$ can be easily obtained from (10) – (12) and have the form:

$$m_1 = 2t^2ab + t^4ab^3 - t^2am_2 - t^4m_2ab^2 + \frac{t^4}{4}m_2ab - 4t^4ab^2q - 2t^4abq^2 + 2t^4abm_2q, \quad (27)$$

$$q = 4t^2b^2q + 4t^4a^2b^2 - 4t^4ba^2m_2 + t^4a^2m_2^2 + 8t^4b^4q - 4t^4b^3qm_2 + 2t^4q^2(a^2 - 8b^2), \quad (28)$$

$$m_2 = -2b^3t^2 + t^2b^2m_2 - t^4b^2(2a^2 - b^2) + t^4bm_2(2a^2 - b^2) - \frac{t^4}{4}(2a^2 - b^2)m_2^2 + 2t^4b^3q^2 + 4t^4a^2bq - 2t^4a^2m_2q. \quad (29)$$

Here

$$a = 3\eta^2 - 1, \quad b = 1 + \eta^2.$$

If $\eta = 1/\sqrt{3}$ the Hamiltonian becomes analogous to the $S = 1$ spin glass Hamiltonian. There is the trivial solution $q = 0$ when $T > T_c^b$ because $A_3 = 0$. The solution $q \neq 0$ appears at the point $kT_c^b/\tilde{J} \approx 3.2$ (for $m = 0$) [11]. The specific heat function loses smoothness at this point (see fig.1). In the case $a \neq 0$ there is no trivial solution of Eq. (28) and q , m_2 and m_1 grows continuously on cooling. This case is analogous to quadrupolar glass with two internal fields. The absence of the zero solution at high temperatures for glass order parameter is unambiguously connected with the absence of reflection symmetry.

In the figures we present the behavior of the order parameters for different cases as well as the specific heat

$$\frac{C_v}{kN} = \frac{d}{d(kT/\tilde{J})} \left\{ \left(\frac{\tilde{J}}{kT} \right) \frac{(q^2 - p^2)}{2} \right\} - \left(\frac{\tilde{J}_0}{\tilde{J}} \right) m_1 \frac{dm_1}{d(kT/\tilde{J})} \quad (30)$$

Let us note that the symmetry rule for the type of the growing of glass regime and phase transition in pure

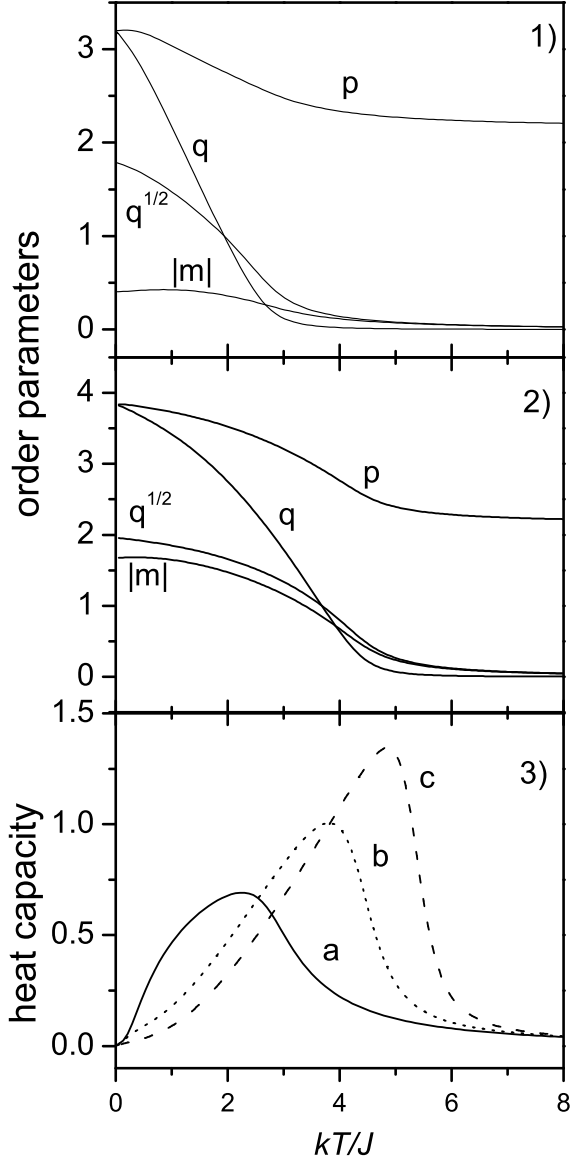


Fig. 1. Order parameters for $\eta = 0.5/\sqrt{3}$ 1) $\tilde{J}_0 = 0$; 2) $\tilde{J}_0/\tilde{J} = 1.5$; and 3) specific heat for a) $\tilde{J}_0/\tilde{J} = 0$; b) $\tilde{J}_0/\tilde{J} = 1.5$; c) $\tilde{J}_0/\tilde{J} = 2$.

systems have been obtained in MFA. It is interesting to notice that in the Bethe approximation (and in some other cluster approximations) the internal fields appear in the case of nonzero cubic terms and cause a smooth changing of the order parameter without phase transition in non-random system if in MFA the phase transition is of the first order (see, e.g., [14]). In the systems with reflection symmetry and the MFA second order phase transition the rather sharp phase transition from nonzero order parameter phase remains in the Bethe approximation. In some way the replica approach to random systems effects analogously to Bethe approximation.

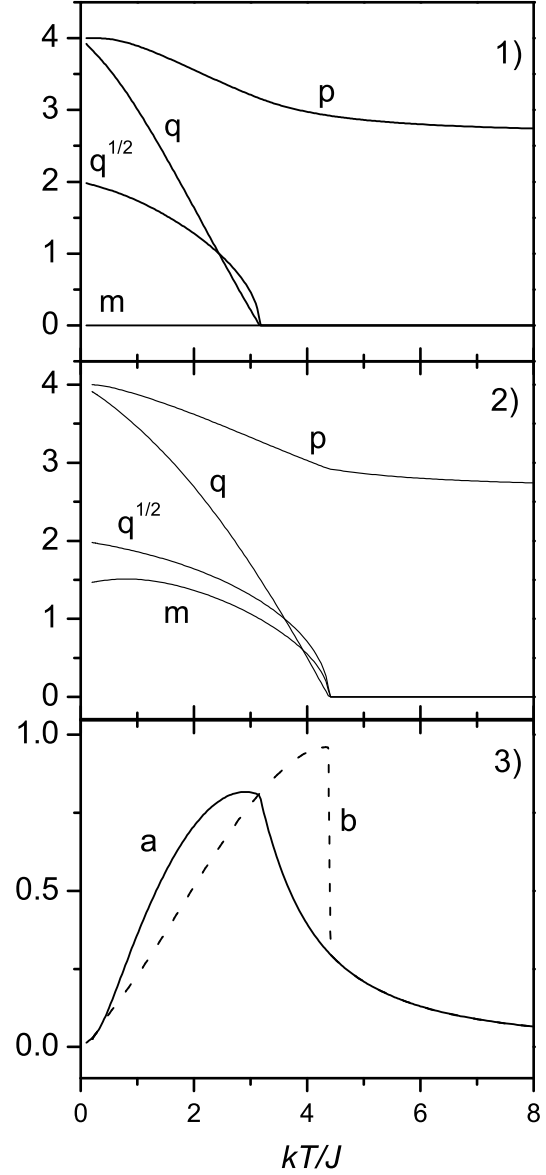


Fig. 2. Order parameters for $\eta = 1/\sqrt{3}$ 1) $\tilde{J}_0 = 0$; 2) $\tilde{J}_0/\tilde{J} = 1.5$; and 3) specific heat for a) $\tilde{J}_0/\tilde{J} = 0$; b) $\tilde{J}_0/\tilde{J} = 1.5$.

The simplest way to demonstrate this fact is to consider cluster approximation of [16]. The main equation is the selfconsistency condition given in terms of effective mean field ψ produced by the neighbors and acting on the particle on each site of the cluster.

For the Hamiltonian (1) in nonrandom case with nearest neighbor interaction we obtain the following equations

[15,16]:

$$\langle \hat{U} \rangle = \frac{\text{Tr} \hat{U}_1 \exp \left[-\beta \hat{H}_1(\psi) \right]}{\text{Tr} \exp \left[-\beta \hat{H}_1(\psi) \right]} = \frac{\text{Tr} \hat{U}_1 \exp \left[-\beta \hat{H}_s(\psi) \right]}{\text{Tr} \exp \left[-\beta \hat{H}_s(\psi) \right]} \quad (31)$$

where s is the number of particles in the cluster. For the simplest case $s = 2$ we have where $H_1 = -\hat{U}_1\psi$; $H_2 = -J\hat{U}_1\hat{U}_2 - \psi(1 - 1/\gamma)(\hat{U}_1 + \hat{U}_2)$; \hat{U}_i is the operator \hat{U} on i -th site; γ is the number of nearest neighbors. The effective mean field ψ can be obtained from (31). The high temperature expansion for (31) is

$$\begin{aligned} \psi\beta J \text{Tr}(\hat{U}^2) + \psi^2 \frac{\beta^2 J^2}{2} \text{Tr}(\hat{U}^3) &= \psi\alpha\beta \text{Tr}(\hat{U}^2) + \\ \frac{\beta^2 J^2}{2} \text{Tr}(\hat{U}^2) \text{Tr}(\hat{U}^3) + \psi^2 \frac{\alpha^2 \beta^2}{2} \text{Tr}(\hat{U}^3) &+ \\ + \psi\beta^2 \alpha \left[\text{Tr}(\hat{U}^2) \right]^2, \end{aligned} \quad (32)$$

where $\alpha = 1 - 1/\gamma$.

It is easy to see from (32) that there is no solution $\psi = 0$ for the case $\text{Tr}(\hat{U}^3) \neq 0$. So there is no phase transition and the order parameter grows smoothly on cooling. When $\text{Tr}(\hat{U}^3) = 0$ a bifurcation point can exist for Eq. (32) and in the pure system we obtain a phase transition.

In conclusion we formulate a kind of symmetry rule for the way of the appearance of glass regime: if in the nonrandom (pure) system the transition to the ordered phase is of the second order, then in the corresponding random system the glass regime appears as a result of a phase transition; if the transition in the pure system is of the first order, then in the random system the glass regime grows continuously on cooling [8].

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